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# TORSIONAL-FLEXURAL BUCKLING OF THIN-WALL COLUMN OF SINGLE SYMMETRICAL OPEN SECTION USING RITZ METHOD

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*Abstract:* **This paper investigates the torsional-flexural buckling behavior of thin-walled columns with single symmetric open sections. It focuses on deriving and solving the buckling equations for these specific column configurations. The analysis reduces the problem to a system of algebraic eigenvalue-eigenvector problems, identifying the critical buckling loads and modes. The buckling behavior is described by a system of three homogeneous differential equations, with two uncoupled equations, simplifying the analysis. Numerical examples illustrate that critical buckling loads decrease as column length increases, highlighting the relationship between length and stability. The results were validated through comparisons with established methods, including the differential equations method by Jerath (2020) and the equilibrium of deformed shape approach by Iyengar (1988), both of which show consistent results. This research contributes to a deeper understanding of the stability of thin-walled columns, providing essential insights for structural design and safety.**

*Keywords:* **Single symmetric section, thin-walled column, flexural-torsional buckling, Ritz method, eigenvalue-eigenvector problem.**

## 1. INTRODUCTION

Thin-walled structures find diverse engineering applications across industries such as aerospace, automotive, and construction, proving pivotal in reducing overall structure's weight. Among systems designed to prevent buckling efficiently, thin-walled members stand out due to their optimal material usage and composition of several thin parts (Al-Ansari, Abdulsamad, Gburi, & Al-Anssari, 2020). These parts allow easy formation into various shapes with high shape factor, minimizing material consumption.

However, thin-wall columns are susceptible to buckling, which is the sudden lateral deflection or failure of the column under compressive loads (Bin, Mohamed, Aabid, & Ibrahim, 2022). Buckling is a common phenomenon observed in thin-walled structures, referring to the loss of stability in a component due to lateral deflection when subjected to an axial force.

The column's weakness causes it to bend, leading to rapid and potentially hazardous failure. Whether a column buckles or not depends on its length, strength, and other relevant factors. According to Baird, Hendy, Wong, Jones, Sollis, and Nuttall (2011), elastic buckling is more likely to occur in long columns relative to their thickness or when an applied compressive load surpasses the critical allowable load of the thin-walled structure. In general, the buckling failure usually occurs in thin-walled open cross-sections due to a combination of torsion and bending Jerath, (2020).The buckling mode represents the shape or pattern of the deformation that occurs during buckling.

The buckling behavior depends on the column's slenderness ratio and boundary conditions. Proper analysis and design techniques, such as incorporating bracing or using appropriate reinforcement, are employed to prevent buckling and ensure the stability and strength of thin-wall columns.

Subsequent work by Wagner (1929) and later work by Bleich (1952) and also by Timoshenko and Gere (1961) led to the development of a general theory of flexural-torsional buckling. They provided the classical energy equation for calculating the elastic flexural-torisonal buckling loads of thin-walled beams. Galambos (1963) introduced inelastic behavior of the flexural-torsional buckling; similar research was also presented by Lee (1960), White (1956), Wittrick (1952), and Horne (1950). All of these researches were done using the classical method, which provided exact solutions, yet it is limited by the necessity to make extensive calculations by hand. This situation changed dramatically with the advent of digital computers in the 1960's.

The classical energy equations for calculating the elastic flexural-torsional buckling load of thin-walled beams are usually assumed to be independent of the prebuckling deflections. The strain energy stored throughout deformation, U is obtained by the product of the strain components and their corresponding stress components integrated over the volume of the elastic body. (Arizou, 2020).

Iyengar (1988) presented a work on torsional-flexural buckling of open section using equilibrium method. More recent research on the theory of flexural-torsional buckling has been presented by Tong and Zhang (2003a) and (2003b) with their investigations of a new theory to clarify the inconsistencies of existing theories of the flexural-torsional buckling of thin-walled members. Ezeh (2009) conducted a theoretical analysis based on Vlasov's theory, as modified by Varbanov, to examine flexural, flexural-torsional, and flexural-torsional-distortional buckling modes of thin-walled closed columns. Chidolue and Osadebe (2012) also employed Vlasov's theory for the torsional-distortional analysis of thin-walled box girder bridges. Similarly, Chidolue and Aginam (2012) studied the effects of shape factor on the flexural-torsional-distortional behavior of thin-walled box girder structures using Vlasov's theory. In another study, Ezeh (2010) investigated the buckling behavior of axially compressed multi-cell doubly symmetric thin-walled columns using Vlasov's theory. Additional works by Osadebe and Chidolue (2012a, 2012b), and Osadebe and Ezeh (2009a, 2009b) were also grounded in Vlasov's method. Furthermore, Ezeh and Osadebe (2010) conducted a comparative study on Vlasov and Euler instabilities of axially compressed thin-walled box columns.

## 2. THEORETICAL FRAMEWORK

Since thin-walled structures tend to be slender, they are vulnerable to buckling instabilities at the local as well as the global scales. Therefore it is necessary to derive simplified stability equation for flexural-torsional (FT) buckling analysis of thin-walled columns of open sections. This research aims to achieve the following objectives:

i. To determine the total potential energy functional of a thin-walled column with single symmetric open cross-section undergoing flexural-torsional buckling.

ii. To obtain the differential equation for the flexural-torsional buckling analysis of single symmetric thin-walled columns with open cross-sections.

iii. To obtain elastic buckling equation using energy formulation for single symmetric thin-walled columns with open cross-sections.

iv. To solve numerical problems with the method developed herein

### 2.1 Assumptions

The energy formulation is based on the following assumptions:

(i) Shear centre of the cross-section is chosen as the origin.

(ii) The x and y coordinate axes are assumed to be coincident with the principal axes of the open cross-section, and the z coordinate axis is the longitudinal axis of the thin-walled column through the shear centre.

(iii) The displacement field include the displacements along x, y and z direction designated as u, v and w. Strain energy and external work for each case shall be treated independently first. The positions O and S stand for the centroid and shear center respectively.

(iv) The displacement of the shear center along y-axis and z- axis are denoted as v and w respectively. On the other hand, the displacement of the centroid along y-axis and z-axisare denoted as v*. w*.

(v) The linear space between the shear center and the centroid remains constant after the translational displacement.

(vi) The shear centre of the cross-section is chosen as the origin. The x and y coordinate axes are assumed to be coincident with the principal axes of the open cross-section, and the z coordinate axis is the longitudinal axis of the thin-walled column through the shear centre.

**2.2 Determination of the Total Potential Energy Functional for single symmetric Thin-Walled Column with Open Cross-Section undergoing Flexural-Torsional Buckling**

The total potential energy functional (∏) for the single symmetric thin-walled column with open cross-section under flexural-torsional buckling is the sum of the strain energy functional U and the potential energy due to the external compressive load V

$$\Pi = U - V \quad (1)$$

Consider a column with arbitrary cross section shown In Figure 2.1.

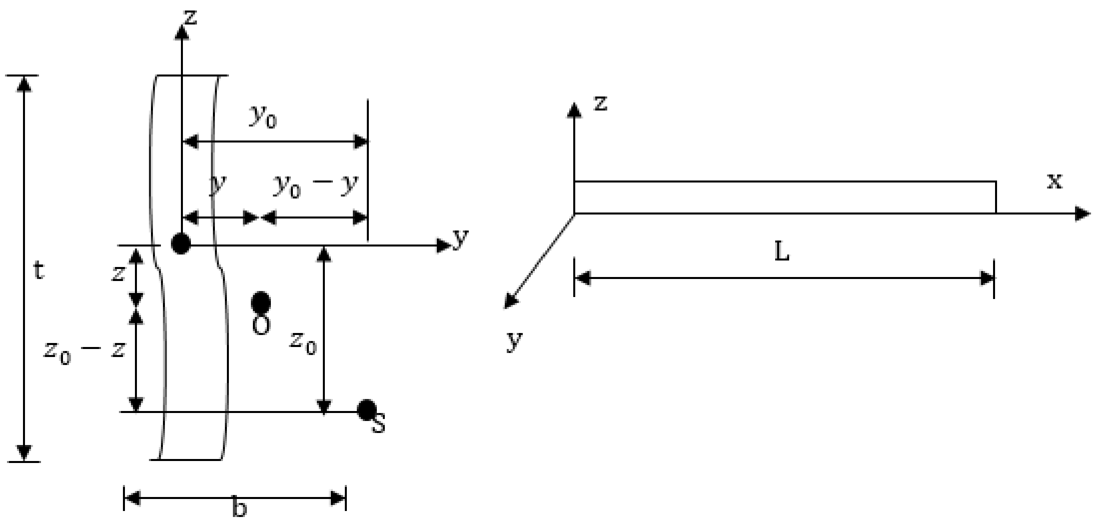


**Figure 2.1: Column under axial load**

This analysis comprises of column buckling under the following cases:

i. Pure flexural buckling

ii. Pure torsional buckling

iii. Flexure - torsional buckling

$$w^* = w - (y_0 - y).\emptyset \quad (2)$$

$$v^* = v + (z_0 - z).\emptyset \quad (3)$$

Where ∅ rotation of the cross-section about the shear is center O, $y_0$ and $z_0$represent the coordinates of the shear center O

**2.2.1 Flexural Buckling**

Let a portion of the column be considered.

The total change in length of the column after buckling is given by equation (4).

That is:

$$\Delta_z = \frac{1}{2}\int_0^L \left(\frac{dw^*}{dx}\right)^2 dx \tag{4}$$

If the buckling occurred in the y direction, the change in length shall be obtained by modifying Equation 4 appropriately as:

$$\Delta_y = \frac{1}{2}\int_0^L \left(\frac{dv^*}{dx}\right)^2 dx \tag{5}$$

Total buckling is obtained by adding the buckling in both y and z directions. That is adding Equations (1) and (5), which gives

$$\Delta = \frac{1}{2}\int_0^L \left[\left(\frac{dv^*}{dx}\right)^2 + \left(\frac{dw^*}{dx}\right)^2\right] dx \tag{6}$$

The indefinite summation product of axial stress and buckling caused by it within the domain (cross section area of the column) gives the external work:

$$V = \iint_A \sigma_x \Delta \, dA \tag{7}$$

From Kirchhoff's assumptions of zero shear strains:

$$\gamma_{xz} = \frac{du_z}{dz} + \frac{dw}{dx} = 0 \tag{8}$$

$$\gamma_{xy} = \frac{du_y}{dy} + \frac{dv}{dx} = 0 \tag{9}$$

Solving Equations (8) and (9), yields Equation (10) and (11) respectively:

$$u_z = -z\frac{dw}{dx} \tag{10}$$

$$u_y = -y\frac{dv}{dx} \tag{11}$$

Normal strain in x direction is the first derivative of Equations (6) and (7) with respect to x:

$$\varepsilon_x{}^z = -z\frac{d^2w}{dx^2} \tag{12}$$

$$\varepsilon_x{}^z = -y\frac{d^2v}{dx^2} \tag{13}$$

Adding Equations (12) and (13) gives the normal strain along x-axis as:

$$\varepsilon_x = -\left(z\frac{d^2w}{dx^2} + y\frac{d^2v}{dx^2}\right) \tag{14}$$

From Hooke's law, stress is mathematically defined by Equation (15)

$$\sigma_x = E\varepsilon_x = -E\left(z\frac{d^2w}{dx^2} + y\frac{d^2v}{dx^2}\right) \tag{15}$$

Average strain energy is given by Equation (16)

$$U = \frac{E}{2}\int \left[I_z\left(\frac{d^2w}{dx^2}\right)^2 + 2I_{yz}\frac{d^2w}{dx^2} * \frac{d^2v}{dx^2} + I_y\left(\frac{d^2v}{dx^2}\right)^2\right] dx \tag{16}$$

Where:

$$I_{yz} = \int_{b_1}^{b_2}\int_{t_1}^{t_2} yz\, dydz = 0 \tag{17}$$

Substituting Equation (17) into Equation (16) gives:

$$U = \frac{E}{2}\int\left[I_z\left(\frac{d^2w}{dx^2}\right)^2 + I_y\left(\frac{d^2v}{dx^2}\right)^2\right]dx \tag{18a}$$

Substituting Equation (15) into Equation (7) gives:

$$V = \iint_A \sigma_x \frac{1}{2}\int_0^L\left[\left(\frac{dv^*}{dx}\right)^2 + \left(\frac{dw^*}{dx}\right)^2\right]dx\, dA \tag{18b}$$

Where A is cross sectional area

That is:

$$V = \frac{\sigma_x}{2}\iint_A \int_0^L\left(\left[\frac{dv}{dx}\right]^2 + \left[\frac{dw}{dx}\right]^2 + 2[z_0 - z].\frac{dv}{dx}\frac{d\emptyset}{dx} - 2[y_0 - y].\frac{dw}{dx}\frac{d\emptyset}{dx} + [{y_0}^2 - 2y_0y + y^2 + {z_0}^2 - 2z_0z + z^2].\left[\frac{d\emptyset}{dx}\right]^2\right)dx\, dA \tag{19}$$

If a column section is symmetrical about two axes, the shear center coincides with the centroid, and we have $y_0 = z_0 = 0$.

That is:

$$V = \frac{\sigma_x A}{2}\int_0^L\left(\left[\frac{dv}{dx}\right]^2 + \left[\frac{dw}{dx}\right]^2 + \left[{y_0}^2 + {z_0}^2 + \frac{I_y + I_z}{A}\right].\left[\frac{d\emptyset}{dx}\right]^2 - 2y_0.\frac{dw}{dx}\frac{d\emptyset}{dx} + 2z_0.\frac{dv}{dx}\frac{d\emptyset}{dx}\right)dx \tag{20}$$

Where:

$$I_y = \iint_A y^2\ dA \tag{21}$$

$$I_z = \iint_A z^2\ dA \tag{22}$$

$$0 = \iint_A y\ dA = \iint_A z\ dA \tag{23}$$

Since the case of pure flexural buckling is considered, the torsional work is ignored. Thus, Equation (20) becomes:

$$V = \frac{N_x}{2}\int_0^L\left[\left[\frac{dv}{dx}\right]^2 + \left[\frac{dw}{dx}\right]^2\right]dx \tag{24}$$

Where:

$$I_0 = A{y_0}^2 + A{z_0}^2 + I_y + I_z \tag{25}$$

$$N_x = \sigma_x A \tag{26}$$

Total potential energy is the algebraic summation of strain energy and external work:

$$\Pi = U - V \tag{27}$$

Substituting Equation (20) and (24) into Equation (27) gives:

$$\Pi = \frac{EI_z}{2}\int\left(\frac{d^2w}{dx^2}\right)^2 dx + \frac{EI_y}{2}\int\left(\frac{d^2v}{dx^2}\right)^2 dx - \frac{N_x}{2}\int_0^L\left(\frac{dw}{dx}\right)^2 dx - \frac{N_x}{2}\int_0^L\left(\frac{dv}{dx}\right)^2 dx \tag{28}$$

### 2.2.2 Total Potential Energy Functional for Thin-Walled Column of Open Cross-Section In Torsional Buckling

This case has two sub cases. Case A is a case where ends are allowed to warp, and sub case B, where ends are prevented from warping.

#### 2.2.2.1 Case A: End Free to Warp

Assume the ends of the column in Figure 2.1 are allowed to warp as illustrated on Figure 2.2:

From similar triangles:

$$\frac{v_1}{(t_1)} = \frac{v_2}{(t_2)} = \tan\phi \qquad (29)$$

$$\frac{w_1}{(b_1)} = \frac{w_2}{(b_2)} = tan\,\phi \qquad (30)$$

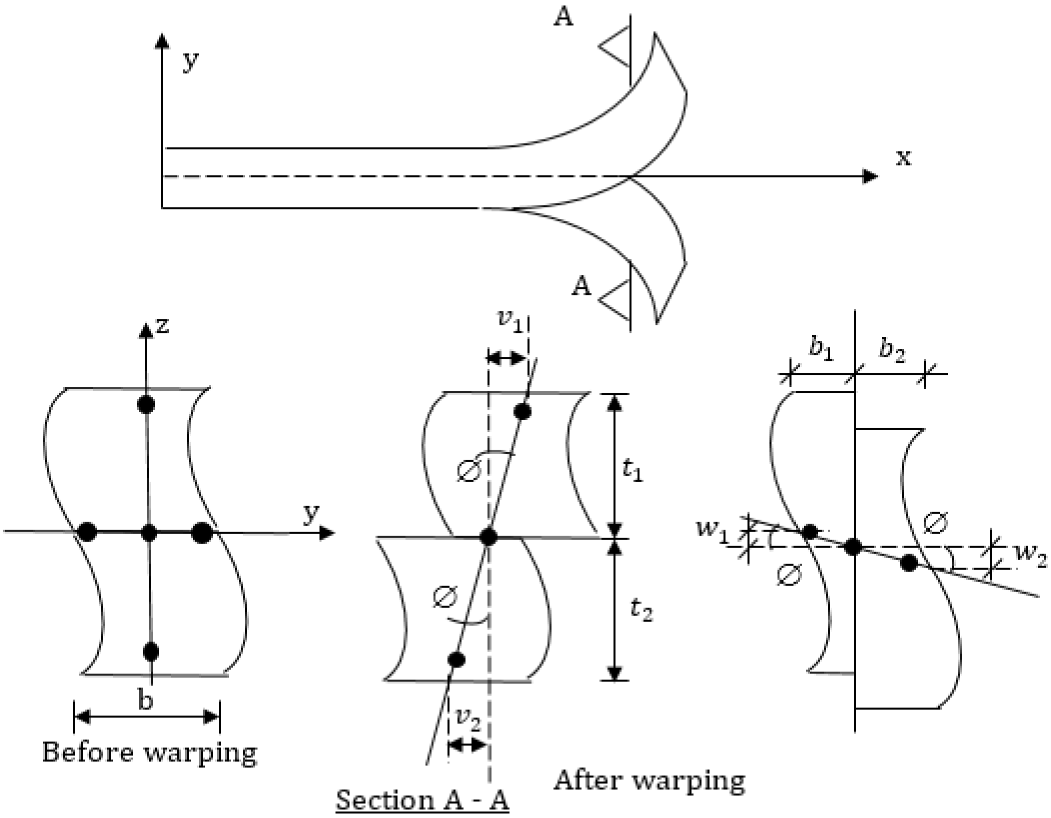


**Figure 2.2: One end of column warped**

$$\phi = \tan\phi \qquad (31)$$

For small deformation, substituting Equation (31) into Equations (29) and (30) gives respectively:

$$v_i = \mathrm{t}_i\,\phi \qquad (32)$$

$$w_i = \mathrm{b}_i\,\phi \qquad (33)$$

Equations (32) and (33) are rewritten as Equation (34) and Equation (35) respectively

$$v = z\,\phi \qquad (34)$$

$$w = y\,\phi \qquad (35)$$

From Kirchhoff's assumptions of zero shear strains, Equation (36) is obtain

$$\gamma_{xy} = \frac{du_y}{dy} + \frac{dv}{dx} = 0 \qquad (36)$$

Solving Equation (36) gives:

$$u_y = -y\frac{dv}{dx} \tag{37}$$

Substituting Equation (32) into Equation (37) gives:

$$u_y = -y\,\mathrm{t}_i\frac{d\phi}{dx} \tag{38}$$

Similarly if the warping of the flanges moves in z direction, then:

$$u_z = -\mathrm{zb}_i\frac{d\phi}{dx} \tag{39}$$

Axial displacement is obtained by the summation of Equations (38) and (39). That is:

$$u = u_y + u_z = -y\,\mathrm{t}_i\frac{d\phi}{dx} - \mathrm{zb}_i\frac{d\phi}{dx} = -(y\,\mathrm{t}_i + \mathrm{zb}_i)\frac{d\phi}{dx} \tag{40}$$

Normal strain in x direction is the first derivative of Equations (40) with respect to x:

$$\varepsilon_x = -(y\,\mathrm{t}_i + \mathrm{zb}_i)\frac{d^2\phi}{dx^2} \tag{41}$$

The twist (shear strain around x-axis) is obtained by adding the first derivatives of w and v with respect to x. That is the summation of the first derivatives of Equations (34) and (35) with respect to x yields Equation (42)

$$\gamma_s = \varepsilon_{zx} + \varepsilon_{yx} \tag{42}$$

Where $\gamma_s$ is shear strain around x-axis

The first derivatives of Equation (34) and (35) with respect to x are:

$$\varepsilon_{yx} = \frac{dv}{dx} = z\frac{d\phi}{dx} \tag{43}$$

$$\varepsilon_{zx} = \frac{dw}{dx} = y\frac{d\phi}{dx} \tag{44}$$

Substituting Equations (43) and (44) into (42) gives:

$$\gamma_s = (y+z)\frac{d\phi}{dx} \tag{45}$$

From Hooke's law, normal and shear stresses are mathematically defined by Equation (46) and (47) respectively:

$$\sigma_x = E\varepsilon_x = -E(y\,\mathrm{t}_i + \mathrm{zb}_i)\frac{d^2\phi}{dx^2} \tag{46}$$

$$\tau_s = G\gamma_s = G(y+z)\frac{d\phi}{dx} \tag{47}$$

Where $\tau_s$ is shear stress

Average strain energy, $U_y$, is given by Equation (48)

$$U_y = \frac{1}{2}\int\int_{b_1}^{b_2}\int_{t_1}^{t_2}(\sigma_x\varepsilon_x + \tau_s\gamma_s)\,dxdydz \tag{48}$$

Substituting Equations (41), (42), (45) and (46) into Equation (48) gives:

$$U = \frac{EI_\omega}{2}\int\left(\frac{d^2\phi}{dx^2}\right)^2 dx + \frac{GJ}{2}\int\left(\frac{d\phi}{dx}\right)^2 dx \tag{49}$$

Where: the warping torsional constant $I_w$ and St. Venant torsional constant J are defined by Equation (50) and (51) respectively

$$I_\omega = \int_{b_1}^{b_2} \int_{t_1}^{t_2} (y\,\mathrm{t}_i + z\mathrm{b}_i)^2 \, dydz \tag{50}$$

$$J = \int_{b_1}^{b_2} \int_{t_1}^{t_2} (y+z)^2 \, dydz \tag{51}$$

Rearranging Equation (11) and squaring both sides, yields Equation (52):

$$\left(\frac{u}{y}\right)^2 = \left(\frac{dv}{dx}\right)^2 \tag{52}$$

Substituting Equation (52) into Equation (20) gives Equation (53)

$$V_y = \frac{1}{2}\int_0^L \int_{t_1}^{t_2} N_x \left(\frac{u}{y}\right)^2 dxdz \tag{53}$$

Rearranging Equation (38), gives Equation (54)

$$\frac{u}{y} = -\mathrm{t}_i \frac{d\phi}{dx} \tag{54}$$

Substituting Equation (54) into Equation (53), yields Equation (55):

$$V_y = \frac{N}{2}\frac{\mathrm{I}_0}{\mathrm{A_c}}\int_0^L \left(\frac{d\phi}{dx}\right)^2 dx \tag{55}$$

Where:

$$\frac{N\mathrm{I}_0}{\mathrm{A_c}} = \int_{t_1}^{t_2} N_x \mathrm{t}_i^{\,2} \, dz \tag{56}$$

Where: $A_c$ is the area of the cross section and $I_o$ is the polar moment of inertia of the cross-section about the longitudinal axis passing through the shear center O.

Adding equation (49) and (55) gives the following total potential energy:

$$\Pi = \frac{EI_\omega}{2}\int \left(\frac{d^2\phi}{dx^2}\right)^2 dx - \frac{N}{2}\frac{\mathrm{I}_0}{\mathrm{A_c}}\int_0^L \left(\frac{d\phi}{dx}\right)^2 dx \tag{57}$$

#### 2.2.2.2 Case B: Ends Prevented from Warp

This is a case where only twisting is applied with ends prevented from warping. Consider the column in Figure 2.1 assumed to be a circular bar with the cross section shown in Figure 2.3.

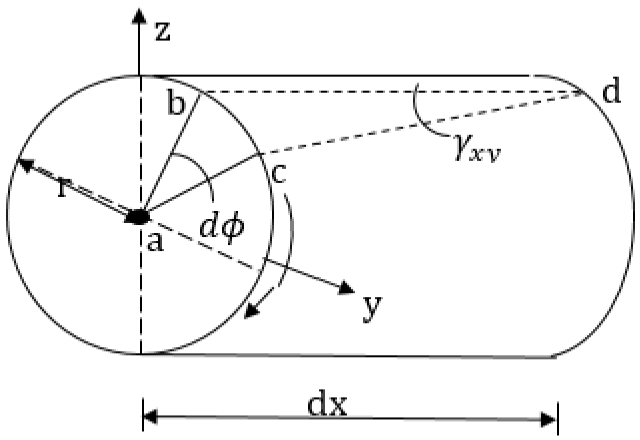


**Figure 2.3: Small length dx of a circular shaft undergoing twisting**

Since the deformation is small, the angular twist and shear strain are defined by Equation (58) and Equation (59) respectively:

$$d\phi = \sin d\phi = \frac{bc}{ac} = \frac{bc}{r} \quad (58)$$

$$\gamma_{xy} = \sin \gamma_{xy} = \frac{bc}{bd} = \frac{bc}{dx} \quad (59)$$

From Equations (58) and (59), obtain Equation (60):

$$\gamma_{xy} = r\frac{d\phi}{dx} \quad (60)$$

Assume r = y, then Equation (60) becomes

$$\gamma_{xy} = y\frac{d\phi}{dx} \quad (61)$$

On the other hand if r be z, then Equation (60) to become Equation (62):

$$\gamma_{xz} = z\frac{d\phi}{dx} \quad (62)$$

From Hooke's law, shear stress is mathematically defined by Equation (63):

$$\tau_{xy} = G\gamma_{xy} \quad (63)$$

Substituting Equation (61) into Equation (63) gives Equation (64):

$$\tau_{xy} = G.y.\ \frac{d\phi}{dx} \quad (64)$$

Where G is shear modulus

But strain energy is defined as the product of stress, strain and volume of matter. Thus, average strain energy is given by Equation (65):

$$U_{xy} = \frac{1}{2}\int\int\left[\int \tau_{xy}\,.\gamma_{xy}\,dy\right]dx\,dz \quad (65)$$

Substituting Equations (61) and (64) into Equation (65) yields Equation (66)

$$U_{xy} = \frac{GJ}{2}\int\left(\frac{d\phi}{dx}\right)^2 dx \quad (66)$$

Where J is the St. Venant torsional constant defined by Equation (67)

$$J = \int\int y^2\,dydz \quad (67)$$

The total potential energy of a column subject to torsional buckling, is obtained by adding equations (57) and (66):

$$\Pi = \frac{EI_\omega}{2}\int\left(\frac{d^2\phi}{dx^2}\right)^2 dx - \frac{N_x}{2}\frac{\mathrm{I}_0}{\mathrm{A_c}}\int_0^L\left(\frac{d\phi}{dx}\right)^2 dx + \frac{GJ}{2}\int\left(\frac{d\phi}{dx}\right)^2 dx \quad (68)$$

**2.2.3 Total Potential Energy Functional for Thin-Walled Column of Open Cross-Section in Flexural-Torsional Buckling**

The strain energy U for this case is obtained by adding Equations (18), (49) and (66):

$$U = \frac{EI_z}{2}\int\left(\frac{d^2w}{dx^2}\right)^2 dx + \frac{EI_y}{2}\int\left(\frac{d^2v}{dx^2}\right)^2 dx + \frac{EI_\omega}{2}\int\left(\frac{d^2\phi}{dx^2}\right)^2 dx + \frac{GJ}{2}\int\left(\frac{d\phi}{dx}\right)^2 dx \quad (69)$$

Where:

$$I_z = \iint_A z^2 \ dA \tag{70}$$

$$I_y = \iint_A y^2 \ dA \tag{71}$$

$$I_\omega = \iint_A (y\,\mathrm{t}_i + z\mathrm{b}_i)^2 \ dA \tag{72}$$

$$J = \iint_A (y+z)^2 \ dA \tag{73}$$

Subtracting Equation (24) from Equation (69) gives the following total potential energy functional Π:

$$\Pi = \frac{E}{2}\int \left[ I_z\left(\frac{d^2w}{dx^2}\right)^2 + I_y\left(\frac{d^2v}{dx^2}\right)^2 + I_\omega\left(\frac{d^2\phi}{dx^2}\right)^2 + \frac{GJ}{E}\left(\frac{d\phi}{dx}\right)^2 - \frac{N_x}{E}\left(\frac{dv}{dx}\right)^2 - \frac{N_x}{E}\left(\frac{dw}{dx}\right)^2 - \frac{N_xI_0}{EA}.\left(\frac{d\emptyset}{dx}\right)^2 + \frac{2y_0N_x}{E}.\frac{dw}{dx}.\frac{d\emptyset}{dx} - \frac{2z_0N_x}{E}.\frac{dv}{dx}.\frac{d\emptyset}{dx}\right] dx \tag{74}$$

Where:

$$I_0 = Ay_0{}^2 + Az_0{}^2 + I_y + I_z \tag{75}$$

Equation (75) can be rewritten in term of non-dimensional coordinate, R (where: R = x/L) as follows:

$$\Pi = \frac{E}{2L^3}\int \left[ I_z\left(\frac{d^2w}{dR^2}\right)^2 + I_y\left(\frac{d^2v}{dR^2}\right)^2 + I_\omega\left(\frac{d^2\phi}{dR^2}\right)^2 + \frac{GJL^2}{E}\left(\frac{d\phi}{dR}\right)^2 - \frac{N_xL^2}{E}\left(\frac{dv}{dR}\right)^2 - \frac{N_xL^2}{E}\left(\frac{dw}{dR}\right)^2 - \frac{N_xI_0L^2}{EA}.\left(\frac{d\emptyset}{dR}\right)^2 + \frac{2y_0N_xL^2}{E}.\frac{dw}{dR}.\frac{d\emptyset}{dR} - \frac{2z_0N_xL^2}{E}.\frac{dv}{dR}.\frac{d\emptyset}{dR}\right] dR \tag{76}$$

### 2.3 Determination of the Differential Equation for the Flexural Torsional Buckling Analysis of Thin-Walled Columns with Open Cross-Sections

The differential equations shall be obtained by minimizing the total potential energy functional with respect to the displacement functions. v, w and ϕ.

Minimizing Equation (76) with respect to v gives Equation (77):

$$\frac{I_yE}{N_xL^2}\left(\frac{d^4v}{dR^4}\right) - \frac{d^2v}{dR^2} - z_0\,.\frac{d^2\emptyset}{dR^2} = 0 \tag{77}$$

Minimizing Equation (77) with respect to w yields Equation (78)

$$\frac{I_zE}{N_xL^2}\frac{d^4w}{dR^4} + \frac{d^2w}{dR^2} + y_0.\frac{d^2\emptyset}{dR^2} = 0 \tag{78}$$

Minimizing Equation (76) with respect to $\phi$ gives Equation (79):

$$\frac{I_\omega E}{N_xL^2}\frac{d^4\emptyset}{dR^4} + \left[\frac{GJ}{N_x} - \frac{I_0}{A}\right]\frac{d^2\emptyset}{dR^2} + y_0\,.\frac{d^2w}{dR^2} - z_0\,.\frac{d^2v}{dR^2} = 0 \tag{79}$$

Solving Equations (77). (78) and (79) simultaneously gave the following relations:

$$\emptyset = g_1.v \tag{80}$$

$$\emptyset = g_2.w \tag{81}$$

$$v = g_3.w \tag{82}$$

Where: $g_1, g_2$ and $g_3$ are constants

Substituting Equations (81) and (82) into Equation (76) gives:

$$\Pi = \frac{EI_T}{2L^3}\int\left[\left(\frac{d^2w}{dR^2}\right)^2 + \left(\frac{dw}{dR}\right)^2 . \frac{N_T L^2}{EI_T}\right]dR \quad (83)$$

Where:

$$I_T = I_z + I_y . {g_3}^2 + I_\omega . {g_2}^2 \quad (84)$$

$$N_T = N_x\left(\frac{GJ}{N_x} . {g_2}^2 - {g_3}^2 - 1 - \frac{I_0}{A} . {g_2}^2 + 2. g_2 y_0 - 2. g_2 g_3 z_0\right) \quad (85)$$

Minimizing Equation (83) with respect to w gives the governing equation of the thin-walled open cross section column undergoing flexural-torsional buckling as Equation (86):

$$\frac{d\Pi}{dw} = \frac{EI_T}{L^3}\int\left[\frac{d^4w}{dR^4} + \frac{d^2w}{dR^2} . \frac{N_T L^2}{EI_T}\right]dR = 0 \quad (86)$$

For Equation (86) to be true, its integrand must be zero. That is:

$$\frac{d^4w}{dR^4} + \frac{d^2w}{dR^2} . \frac{N_T L^2}{EI_T} = 0 \quad (87)$$

**2.4 Determination of Buckling Load Formulae**

The ready solution for Equation (87) is:

$$w = hB_2 \quad (88)$$

Where:

$$B_2 = [a_1 a_2 a_3 a_4]^T \quad (89)$$

$$h = [1 \;\; R \;\; cos\, BR \; sin\, BR] \quad (90)$$

Substituting Equation (88) into Equations (80) and (82) respectively yields:

$$B_3 = g_2 . B_2 \quad (91)$$

$$B_1 = g_3 . B_2 \quad (92)$$

Substituting Equation (88) into Equation (83) gives Equation (93):

$$\Pi = \frac{E}{2L^3}\int\Bigg[\left(\frac{d^2h}{dR^2}\right)^2 \left(I_z . {B_2}^2 + I_y . {g_3}^2{B_2}^2 + I_\omega . {g_2}^2{B_2}^2\right) + \left(\frac{dh}{dR}\right)^2\left(\frac{GJ}{N_x} . {g_2}^2{B_2}^2 - {g_3}^2{B_2}^2 - {B_2}^2 - \frac{I_0}{A} . {g_2}^2{B_2}^2 + 2. g_2{B_2}^2 y_0 - 2. g_2 g_3 {B_2}^2 z_0\right) . \frac{N_x L^2}{E}\Bigg] dR \quad (93)$$

Substituting Equations (91) and (92) into Equation (93) gives Equation (94):

$$\Pi = \frac{E}{2L^3}\Bigg({B_1}^2\left[I_y k_{RR} - \frac{N_x L^2}{E}k_R\right] + {B_2}^2\left[I_z k_{RR} - \frac{N_x L^2}{E}k_R\right] + {B_3}^2\left[I_\omega k_{RR} + \frac{L^2}{E}GJk_R - \frac{N_x L^2}{E} . \frac{I_0}{A}k_R\right] + 2B_2B_3y_0 . \frac{N_x L^2}{E}k_R - 2B_1B_3z_0 . \frac{N_x L^2}{E}k_R\Bigg) \quad (92)$$

**2.4.1 Case of Single Symmetrical Open Section**

Minimizing Equation (94) with respect to $B_1$ gives Equation (95):

$$B_1\left[\frac{EI_y}{L^2}\frac{k_{RR}}{k_R} - N_x\right] - B_3 z_0 \,.\, N_x = 0 \tag{95}$$

Minimizing Equation (94) with respect to $B_2$ yields (96):

$$B_2\left[\frac{EI_z}{L^2}\frac{k_{RR}}{k_R} - N_x\right] + B_3 y_0 \,.\, N_x = 0 \tag{96}$$

Minimizing Equation (94) with respect to $B_3$ gives:

$$B_3\left[\frac{EI_\omega}{L^2}\frac{k_{RR}}{k_R} + GJ - N_x.\frac{I_0}{A}\right] + B_2 y_0 \,.\, N_x - B_1 z_0 \,.\, N_x = 0 \tag{97}$$

For pure flexural buckling case, the flexural critical buckling load is obtained from Equations (95), (96) and (97) as Equation (98):

$$N_c = \frac{EI}{L^2}.\frac{k_{RR}}{k_R} \tag{98}$$

Where: I can be either $I_y, I_z$ or $I_\omega$ as the case may be

The Equation (98) is rewritten for various second moments of area as given by Equation (99), (100) and (101):

$$N_{cy} = \frac{EI_y}{L^2}.\frac{k_{RR}}{k_R} \tag{99}$$

$$N_{cz} = \frac{EI_z}{L^2}.\frac{k_{RR}}{k_R} \tag{100}$$

$$N_{c\omega} = \frac{EI_\omega}{L^2}.\frac{k_{RR}}{k_R} \tag{101}$$

Substituting Equations (99), (100) and (101) into Equations (95), (96) and (97) respectively gives Equation (102), (103), and (104) respectively:

$$B_1[N_{cy} - N_x] - B_3 z_0 \,.\, N_x = 0 \tag{102}$$

$$B_2[N_{cz} - N_x] + B_3 y_0 \,.\, N_x = 0 \tag{103}$$

$$B_3[N_\emptyset - N_x.]\frac{I_0}{A} + B_2 y_0 \,.\, N_x - B_1 z_0 \,.\, N_x = 0 \tag{104}$$

Where:

$$N_\emptyset = (N_{c\omega} + GJ)\frac{A}{I_0} \tag{105}$$

$$N_{c\omega} = \frac{EI_\omega}{L^2}.\frac{k_{RR}}{k_R}; \qquad I_\omega = I_{yz} = I_y.h_i^{\,2} + I_z.b_i^{\,2} = warping\ constant$$

$I_z$ is the second moment of area about (around) y axis and $I_y$ is the second moment of area about (around) z axis.

The Equations (102), (103) and (104) can be put in matrix forms as follows:

$$\begin{bmatrix} N_{cy} - N_x & 0 & -z_0 \,.\, N_x \\ 0 & N_{cz} - N_x & y_0 \,.\, N_x \\ -z_0.N_x & y_0.N_x & [N_\emptyset - N_x.]\frac{I_0}{A} \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = 0 \tag{106}$$

For non-trivial solution, the determinant of Equation (106) must be zero. That is:

$$\begin{vmatrix} N_{cy}-N_x & 0 & -z_0\,.N_x \\ 0 & N_{cz}-N_x & y_0\,.N_x \\ -z_0.N_x & y_0.N_x & [N_\emptyset - N_x.]\dfrac{I_0}{A} \end{vmatrix} = 0 \qquad (107)$$

In this case, it is assumed that the axis of symmetry is y axis. Hence, $z_0 = 0$. Substituting for $z_0$ equals zero into Equation (107) gives Equation (108):

$$\begin{vmatrix} N_{cy}-N_x & 0 & 0 \\ 0 & N_{cz}-N_x & y_0\,.N_x \\ 0 & y_0.N_x & [N_\emptyset - N_x.]\dfrac{I_0}{A} \end{vmatrix} = 0 \qquad (108)$$

Equation (108) can be written as two independent equations as:

$$N_{cy} - N_x = 0 \qquad (109)$$

$$\begin{vmatrix} N_{cz}-N_x & y_0\,.N_x \\ y_0\,.N_x & [N_\emptyset - N_x.]\dfrac{I_0}{A} \end{vmatrix} = 0 \qquad (110)$$

The determinant of Equation (110) is given by Equation (111):

$$D_1 {N_x}^2 + D_2 N_x + D_3 = 0 \qquad (111)$$

Where:

$$D_1 = 1 - {y_0}^2 \frac{A}{I_0} \qquad (112)$$

$$D_2 = -[N_{cz} + N_\emptyset] \qquad (113)$$

$$D_3 = N_\emptyset N_{cz} \qquad (114)$$

Using formula for the roots of quadratic equation, it is obtained that:

$$N_x = \frac{-D_2 \pm \sqrt{{D_2}^2 - 4D_1 D_3}}{2D_1} \qquad (115)$$

Substituting Equations (112), (113) and (114) into Equation (115) gives Equation (116):

$$N_x = \frac{(N_{cz} + N_\emptyset \pm \propto)}{2\left(1 - {y_0}^2 \frac{A}{I_0}\right)} \qquad (116)$$

Where:

$$\propto = \sqrt{{N_{cz}}^2 + {N_\emptyset}^2 + 2N_\emptyset N_{cz}\left(2{y_0}^2\frac{A}{I_0} - 1\right)} \qquad (117)$$

## 3. NUMERICAL EXAMPLE

Determine the Critical buckling load of single symmetric channel section with hinged ends. The requirements for the shape functions are that they must satisfy the geometric boundary conditions. The properties of the channel are tabulated below as obtained from Steel Designer's Manual (7th Ed., 2012).

**Table 3.1 Channel Section Designation 180 x 75 x 20**

| Section Designation | d (mm) | b (mm) | $t_w$(mm) | $t_f$ (mm) | A ($cm^2$) | $I_z$ ($cm^4$) | $I_y$ ($cm^4$) | $y_0$ (cm) | $z_0$ (cm) |
|---|---|---|---|---|---|---|---|---|---|
| 180x75x20 Channel | 180 | 75 | 6.0 | 10.5 | 25.9 | 1370 | 146 | 2.87 | 0 |

## 4. RESULTS AND DISCUSSIONS

**4.1 Results**: The problem given in the example was solved using the derived equations. The results obtained were compared with those obtained by Jerath (2020) and Iyengar (1988) and presented in Table 4.1

**Table 4.1 Comparison of Critical Buckling Load, Nx obtained in this Study with other Results**

| Length, L (m) | Critical Buckling Load, Cr (KN) | | | | | | |
|---|---|---|---|---|---|---|---|
| | Present Study Crp | Jerath (2020), Crj | Iyengar (1988), Cri | Crp-Crj | % diff. (Crp-Crj)100 | Crp-Cri | % diff. (Crp-Cri)100 |
| 1 | 3026.02 | 3026.02 | 3026.02 | 0 | 0 | 0 | 0 |
| 1.25 | 1936.65 | 1936.65 | 1936.65 | 0 | 0 | 0 | 0 |
| 1.5 | 1344.9 | 1344.9 | 1344.9 | 0. | 0. | 0. | 0. |
| 1.75 | 988.09 | 988.09 | 988.09 | 0 | 0 | 0 | 0 |
| 2 | 756.51 | 756.51 | 756.51 | 0 | 0 | 0 | 0 |
| 2.25 | 597.73 | 597.73 | 597.73 | 0 | 0 | 0 | 0 |
| 2.5 | 484.16 | 484.16 | 484.16 | 0 | 0 | 0 | 0 |
| 2.75 | 400.13 | 400.13 | 400.13 | 0 | 0 | 0 | 0 |
| 3 | 336.22 | 336.22 | 336.22 | 0 | 0 | 0 | 0 |
| 3.25 | 286.49 | 286.49 | 286.49 | 0 | 0 | 0 | 0 |
| 3.5 | 247.02 | 247.02 | 247.02 | 0 | 0 | 0 | 0 |
| 3.75 | 215.18 | 215.18 | 215.18 | 0 | 0 | 0 | 0 |
| 4 | 189.13 | 189.13 | 189.13 | 0 | 0 | 0 | 0 |
| 4.25 | 167.53 | 167.53 | 167.53 | 0 | 0 | 0 | 0 |
| 4.5 | 149.43 | 149.43 | 149.43 | 0 | 0 | 0 | 0 |
| 4.75 | 134.12 | 134.12 | 134.12 | 0 | 0 | 0 | 0 |
| 5 | 121.04 | 121.04 | 121.04 | 0 | 0 | 0 | 0 |

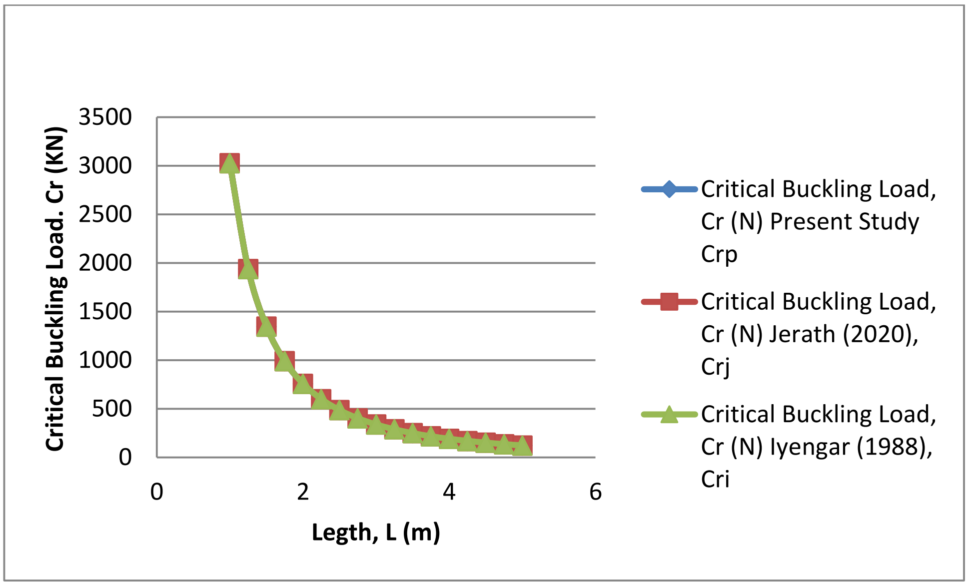


**Figure 4.1: Graph of critical buckling against the length of column for single symmetric channel section**

## 5. DISCUSSION

In solving the problem, the differential equations were found to be reduced to a system of algebraic eigenvalue-eigenvector problems. The buckling equations were derived specifically for single symmetric column sections, identifying the buckling modes as flexural-torsional. For the single symmetric sections, the buckling behavior is described by a system of three homogeneous differential equations, two of which are uncoupled. In the case where the thin-walled column is hinged at both ends, the solution of the one uncoupled buckling equation provides the expression for the critical buckling load in the direction of the axis of symmetry.

The critical buckling loads were found to decrease as the length of the column increases. Comparisons with the corresponding solutions presented by Jerath (2020) using the differential equations method showed identical results, validating the numerical approach used in this study. Additionally, the results were further compared with the solution obtained by Iyengar (1988), which utilized the equilibrium of the deformed shape approach. Both solutions were found to be consistent, reinforcing the reliability of the current study's findings.

## REFERENCES

[1] Al-Ansari, L. S., Abdulsamad, H. J., Gburi, F. H., & Al-Anssari, Q. S. (2020). Analytical and numericalcalculation of the critical buckling load in compound columns. *Australian Journal of Structural Engineering*, *21*(4), 838-854. Available at https://doi.org/10.1080/14484846.2020.1756183

[2] Arizou, R. (2020). *Distortional lateral torsional buckling of doubly symmetric wide flange beams*. A Master's thesis submitted to University of Ottawa.

[3] Baird, D., Hendy, C., Wong, P., Jones, R., Sollis, A., & Nuttall, H. (2011). Design of the Olympic Park bridges H01 and L01. *IABSE Structural Engineering International*, February.

[4] Bin Kamarudin, M. N., Mohamed Ali, J. S., Aabid, A., & Ibrahim, Y. E. (2022). Buckling analysis of a thin-walled structure using finite element method and design ofexperiments. *Aerospace*, *9*(10), 541. Available at https://doi.org/10.3390/aerospace9100541

[5] Bleich, F. (1952). *Buckling strength of metal structures*. McGraw-Hill, New York.

[6] Chidolue, C. A., & Aginam, C. H. (2012). Effects of shape factor on the flexural-torsional distortional behavior of thin-walled box-girder structures. *International Journal of Engineering and Advanced Technology (IJEAT), 1*(5), 469-479.

[7] Chidolue, C. A., & Osadebe, N. N. (2012). Flexural-torsional behavior of thin-walled monosymmetric box-girder structures. *International Journal of Engineering Sciences and Emerging Technologies, 2*(2), 11-19.

[8] Ezeh, J. C. (2009). *Stability of axially compressed multi-cell thin-walled closed column*. A PhD thesis submitted to the Department of Civil Engineering, University of Nigeria, Nsukka.

[9] Ezeh, J. C. (2010). Buckling behavior of axially compressed multi-cell doubly symmetric thin walled column using Vlasov's theory. *International Journal of Engineering*, *4*(2), 179 194.

[10] Ezeh, J. C., & Osadebe, N. N. (2010). Comparative study of Vlasov and Euler instabilities of axially compressed thin-walled box columns. *Nigerian Journal of Technology*, *29*(1), 66 77.

[11] Horne, M. R. (1950). *Critical loading condition of engineering structures* (PhD dissertation).Cambridge University, Cambridge, England.

[12] Iyengar, N. G. R. (1988). *Structural stability of columns and plates* (1st ed.). Ellis Horwood.

[13] Jerath, S. (2020). *Structural stability theory and practice: Buckling of columns, beams, plates,\ and shells* (1st ed.). John Wiley & Sons, Incorporated. ISBN: 9781119694502.

[14] Lee, G. C. (1960). *Literature survey on lateral instability and lateral bracing requirements*. Technical Report.

[15] Osadebe, N. N., & Chidolue, C. A. (2012a). Torsional-distortional response of thin-walled monosymmetric box girder. *International Journal of Engineering Research and Applications, 2*(3), 814-821.

[16] Osadebe, N. N., & Chidolue, C. A. (2012b). Response of double cell monosymmetric box girder structure to torsional-distortional deformations. *International Journal of Engineering and Advanced Technology*, *1*(4), 285-292.

[17] Osadebe, N. N., & Ezeh, J. C. (2009a). Stability of axially compressed single-cell monosymmetric thin-walled closed column. *Nigerian Journal of Technology*, *28*(2).

[18] Osadebe, N. N., & Ezeh, J. C. (2009b). Flexural, torsional, and distortional buckling of single cell thin-walled box columns. *Nigerian Journal of Technology, 28*(2), 80-91.

[19] Piana, G., Lofrano, E., Carpinteri, A., & others. (2021). Effect of local stiffeners and warping constraints on the buckling of symmetric open thin-walled beams with high warping stiffness.*Meccanica, 56*(11), 2083–2102.Available at https://doi.org/10.1007/s11012021-01349-9

[20] Timoshenko, S. P., & Gere, J. M. (1961). *Theory of elastic stability* (2nd ed.). McGraw-Hill Book Company, Inc.

[21] Tong, G., & Lei, Z. (2003a). The transverse stresses in thin-walled beams and its effect on strength and stability. *Advances in Structural Engineering*, *6*, 159–167.

[22] Tong, G., & Lei, Z. (2003b). An analysis of current stability theories for thin-walled members. *Advances in Structural Engineering, 6*(4), 283–292.

[23] Wagner, H. (1929). Torsion and buckling of open sections. In *25th Anniversary Publication,* Technische Hochschule, Danzig.